

Altruism in Networks

Supplementary Appendix

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This supplementary material contains proofs and additional results that complement the paper “Altruism in Networks”. The first section studies the cost minimization problem of the main paper in greater details and uses it to derive additional properties of equilibrium transfers, and provide different proofs of some results. The generic uniqueness of equilibrium transfers is a consequence of this analysis. We also use this section to explain the connections of our game with two classical transportation problems: the minimum cost flow problem, and the Monge-Kantorovich optimal transportation problem. In the second section, we look at best response dynamics in the transfer game, and prove Proposition 2 of the paper.

APPENDIX A : The Cost Minimization Approach

In this section, we analyze in detail the cost minimization problem and use it to draw connection with classical linear programming problems, and to prove the results of the existence and uniqueness section. In particular, we prove Lemma 2 as a corollary of Proposition A.6 in this section, while Lemma 3, Propositions 1, 5, and 7, and Theorems 2 and 3 are direct consequences of Theorem A.2 at the end of this section. We also prove some additional properties of optimal transfer networks such as cyclical monotonicity.

Recall that the maximization of the potential is related to the cost minimization problem

$$c(\mathbf{y}^0, \mathbf{y}) = \min_{\mathbf{T} \in S(\mathbf{y})} \sum_{(i,j) \in A} c_{ij} t_{ij}, \quad (\text{MCF})$$

where $S(\mathbf{y}) = \{\mathbf{T} \in S : \forall i, y_i = y_i^0 - \sum_j t_{ij} + \sum_j t_{ji}\}$ is a closed convex polytope since it is defined by a finite number weak inequalities. Note that $S(\mathbf{y})$ is unbounded if the altruism network A admits a directed cycle, since one can then indefinitely increase the

transfers of any $\mathbf{T} \in S(\mathbf{y})$ along the cycle while still reaching \mathbf{y} from \mathbf{y}^0 . This problem is a classical linear programming problem known as the Minimum Cost Flow problem. Indeed, if each c_{ij} is interpreted as the marginal transportation cost between i and j , this problem consists of minimizing transportation cost over the network of agents with the constraint of reaching distribution \mathbf{y} from distribution \mathbf{y}^0 .

In network flow problems, a transfer profile \mathbf{T} is called a *flow*, and we will sometimes use this terminology. A first useful result from the network flow literature (see Galichon, 2011) says that any flow can be decomposed into paths and cycles. Before we do that, we partition the set of agents into three sets: the sets of net givers $I_G = \{i : y_i < y_i^0\}$ (or *sources*), the set of net receivers $I_R = \{i : y_i > y_i^0\}$ (or *sinks*), and the remaining agents. We let \mathcal{P}_{ij} be the set of paths between i and j in the altruistic network, $\mathcal{P} = \bigcup_{(i,j) \in I_G \times I_R} \mathcal{P}_{ij}$, be the set of paths from net givers to net receivers, and \mathcal{C} be the set of cycles in the network. For any $\rho \in \mathcal{P}$, let h_ρ be the intensity of the flow along ρ , and for each $\gamma \in \mathcal{C}$, let g_γ be the intensity of the flow along γ . A flow on a path is called a path flow, and a flow on a cycle is called a cycle flow. Together, the vectors \mathbf{h} and \mathbf{g} define a feasible transfer profile \mathbf{T} through the equation

$$t_{ij} = \sum_{\rho \in \mathcal{P}} h_\rho \mathbf{1}_{(i,j) \in \rho} + \sum_{\gamma \in \mathcal{C}} g_\gamma \mathbf{1}_{(i,j) \in \gamma}. \quad (1)$$

If in addition,

$$\forall i \in I_G, \sum_{j \in I_R} \sum_{\rho \in \mathcal{P}_{ij}} h_\rho = y_i^0 - y_i, \quad (2)$$

and

$$\forall j \in I_R, \sum_{i \in I_G} \sum_{\rho \in \mathcal{P}_{ij}} h_\rho = y_j - y_j^0, \quad (3)$$

then $\mathbf{T} \in S(\mathbf{y})$. The following proposition shows that every transfer profile can be decomposed in such a way. However, it is easy to see that this decomposition is not necessarily unique. The proof we provide here is adapted from Galichon (2011).

Proposition A.1 (Flow Decomposition) *Any transfer plan $\mathbf{T} \in S(\mathbf{y})$ can be decomposed into path flows and cycle flows of intensities \mathbf{h} and \mathbf{g} according to (1), and such*

that \mathbf{h} satisfies (2) and (3). Conversely any distribution of path flows and cycle flows of intensities \mathbf{h} and \mathbf{g} such that \mathbf{h} satisfies (2) and (3) defines a transfer plan $\mathbf{T} \in S(\mathbf{y})$ through equation (1).

Proof. The second part of the proposition is immediate. For the first part, let $\mathbf{T} \in S(\mathbf{y})$, and consider the following maximization problem

$$\begin{aligned} \max_{\mathbf{h}, \mathbf{g}} \quad & \sum_{\rho \in \mathcal{P}} h_{\rho} + \sum_{\gamma \in \mathcal{C}} g_{\gamma} & (\text{P}) \\ \text{s.t.} \quad & \sum_{\rho \in \mathcal{P}} h_{\rho} \mathbf{1}_{(i,j) \in \rho} + \sum_{\gamma \in \mathcal{C}} g_{\gamma} \mathbf{1}_{(i,j) \in \gamma} \leq t_{i,j}, \quad \forall (i,j) \in A \end{aligned}$$

Because this is a linear program over a bounded set, it has a solution (\mathbf{h}, \mathbf{g}) . Consider the flow \mathbf{T}' , defined by

$$t'_{ij} = \sum_{\rho \in \mathcal{P}} h_{\rho} \mathbf{1}_{(i,j) \in \rho} + \sum_{\gamma \in \mathcal{C}} g_{\gamma} \mathbf{1}_{(i,j) \in \gamma} \leq t_{ij}$$

Suppose that this inequality holds strictly. If $(i, j) \in I_G \times I_R$, then one can increase the flow going through the path $\rho = (i, j) \in \mathcal{P}$ by $t_{ij} - t'_{ij}$ while still satisfying the constraint in (P). Since that would strictly improve the objective function of the program (P), that would lead to a contradiction. Suppose for example that $j \notin I_R$. Then there must exist an agent j' such that $t'_{jj'} < t_{jj'}$, for otherwise the conservation equation at j would be violated by \mathbf{T}' . Similarly, if $i \notin I_G$, then there exists an agent i' such that $t'_{i'i} < t_{i'i}$. Extending t'_{ij} to the left and the right in this way, it must be the case that we end up with a path $\rho \in \mathcal{P}$ that goes through (i, j) , or a cycle $\mu \in \mathcal{C}$ that does not necessarily go through (i, j) , and such that each for each (ℓ, k) that belongs to ρ or μ , $t'_{\ell k} < t_{\ell k}$. Then there is some leeway to increase the intensity h_{ρ} or g_{μ} , and thus strictly improve the objective of the maximization problem (P), while still satisfying its constraint: a contradiction. ■

Note that if \mathbf{T} has a cycle, it may admit a flow decomposition that puts 0 intensity on all cycles. However, there must be a decomposition with a cycle, as the following result shows.

Lemma A.1 \mathbf{T} has a cycle if and only if it admits a flow decomposition that puts positive intensity on a cycle.

Proof. Let $\gamma = (i_0, \dots, i_\ell)$ be a cycle of \mathbf{T} , and let $\tau = \min_{(i,j) \in \gamma} t_{ij} > 0$. Then let \mathbf{g} be the cyclic flow that puts intensity $g_\gamma = \tau$ on the cycle γ , and let $\mathbf{T}' = \mathbf{T} - \mathbf{g}$. \mathbf{T}' is a feasible transfer plan that achieves the same distribution as \mathbf{T} , and therefore it has a flow decomposition $(\mathbf{h}', \mathbf{g}')$. Then $(\mathbf{h}', \mathbf{g}' + \mathbf{g})$ is a flow decomposition of \mathbf{T} that puts positive weight on a cycle. The other implication is trivial. ■

We now introduce a different cost minimization problem, known as the Monge-Kantorovich optimal transportation problem, and show how it is related to the initial problem. In the process, we also prove several important properties of the solutions to (MCF). We start by defining the reduced cost vector \hat{c} with elements

$$\hat{c}_{ij} = \min_{\rho \in \mathcal{P}_{ij}} \sum_{(\ell,k) \in \rho} c_{\ell k},$$

for every $(i, j) \in A$. This reduced cost vector is the cost vector associated with the transitive closure $\hat{\alpha}$ of the altruism network. The cost \hat{c}_{ij} is the lowest cost path between i and j . We call such paths *shortest paths*. They correspond to highest altruism paths. Let \hat{P}_{ij} be the set of shortest paths between i and j .

For every $i \in I_G$ let the $G_i = y_i^0 - y_i$ be the amount of money that needs to be transferred away from i , and for every $j \in I_R$, let $R_j = y_j - y_j^0$ be the amount of money that needs to be transferred to j . By construction, $\sum_{i \in I_G} G_i = \sum_{j \in I_R} R_j$. We can view our problem as that of transferring the amount $\sum_{i \in I_G} G_i$ from I_G to I_R in the least costly way. It is natural to express the cost of transportation between $i \in I_G$ and $j \in I_R$ as \hat{c}_{ij} . Formally,

$$\begin{aligned} & \min_{\tau \in \mathbb{R}_+^{I_G \times I_R}} \sum_{(i,j) \in I_G \times I_R} \hat{c}_{ij} \tau_{ij} & \text{(MK)} \\ \text{s.t.} & \sum_{j \in I_R} \tau_{ij} = G_i, \quad \forall i \in I_G \\ & \sum_{i \in I_G} \tau_{ij} = R_j, \quad \forall j \in I_R. \end{aligned}$$

This program is a Monge-Kantorovich optimal transportation problem with discrete source and target distributions. The two problems are related in the following way. Here again, our presentation borrows from Galichon (2011).

Theorem A.1 *If \mathbf{T} solves (MCF), then it has no cycles, and all the paths with positive intensity in its flow decomposition are shortest paths. Furthermore, the set of solutions to (MCF) is a nonempty, compact and convex polytope. The value $c(\mathbf{y}, \mathbf{y}^0)$ of the cost minimization problem is equal to the value function of (MK). The solutions of (MCF) can be obtained from the solutions of (MK) by distributing each τ_{ij} across the paths in $\hat{\mathcal{P}}_{ij}$, the shortest paths from i to j . The solutions of (MK) can be obtained from the solutions of (MCF) by setting τ_{ij} equal to the sum of the intensities over paths in \mathcal{P}_{ij} in the flow decomposition of a solution \mathbf{T} .*

Proof. Using the flow decomposition theorem, we can rewrite (MCF) as

$$\begin{aligned} \min_{\mathbf{h}, \mathbf{g}} \quad & \sum_{\rho \in \mathcal{P}} h_{\rho} c_{\rho} + \sum_{\gamma \in \mathcal{C}} g_{\gamma} c_{\gamma} \\ \text{s.t.} \quad & \sum_{j \in I_R} \sum_{\rho \in \mathcal{P}_{ij}} h_{\rho} = G_i, \quad \forall i \in I_G \\ & \sum_{i \in I_G} \sum_{\rho \in \mathcal{P}_{ij}} h_{\rho} = R_j, \quad \forall j \in I_R, \end{aligned}$$

where $c_{\rho} = \sum_{(i,j) \in \rho} c_{ij}$ and $c_{\gamma} = \sum_{(i,j) \in \gamma} c_{ij}$.

Since cycles cannot help satisfying the constraints, it is optimal to set $g_{\gamma} = 0$ for every $\gamma \in \mathcal{C}$. Hence optimal transfer networks have no cycle in their flow decomposition, and are therefore acyclic by Lemma A.1. It is also clear that only shortest paths can have strictly positive intensity. Indeed, if there exists a path $\rho \in \mathcal{P}_{ij}$ such that $h_{\rho} > 0$, and ρ is not a shortest path, then reassigning intensity h_{ρ} to another path $\rho' \in \hat{\mathcal{P}}_{ij}$ would lead to a cost reduction of $(c_{\rho} - \hat{c}_{ij})h_{\rho} > 0$.

Having proved that the set of solutions to (MCF) is acyclic, we can solve the minimization problem over the set of acyclic transfer plans. Contrary to the set of transfer plans, it is bounded as no transfer t_{ij} can exceed the total amount of money available $\sum_i y_i^0$. It is

easy to show that it is also a closed set, therefore the minimization problem minimizes a continuous function over a compact set, implying the existence of a solution. Since (MCF) is a linear problem, we know that the solution set is a closed convex polytope, and since all solutions are acyclic, it is also bounded and hence compact.

Since optimal transfers only use shortest paths, we can rewrite the objective function of the transformed program as

$$\sum_{(i,j) \in I_G \times I_R} \sum_{\rho \in \hat{\mathcal{P}}_{ij}} h_\rho c_\rho = \sum_{(i,j) \in I_G \times I_R} \hat{c}_{ij} \sum_{\rho \in \hat{\mathcal{P}}_{ij}} h_\rho.$$

Letting $\tau_{ij} = \sum_{\rho \in \hat{\mathcal{P}}_{ij}} h_\rho$, the transformed program becomes (MCF). This shows that the two programs have the same value function, and how to obtain the solutions of (MCF) and (MK) from one another. ■

To get a better understanding of the structure of the set of solutions to (MCF), we start by describing the structure of the set $S(\mathbf{y})$. We know that it is a possibly unbounded convex polytope. Therefore it can be expressed as the convex hull of a finite set of points and directions. We will now characterize its set of extreme points and directions. The directions will be given by the cycles of A . For every cycle $\gamma \in \mathcal{C}$, let \mathbf{T}^γ be the flow defined by $t_{ij}^\gamma = \mathbf{1}_{(i,j) \in \gamma}$. To describe the set of extreme points, we need some additional notations. Let $\mathbf{T} \in S(\mathbf{y})$ be an acyclic transfer network, so that any flow decomposition of \mathbf{T} is given by a vector \mathbf{h} . Pick any such decomposition \mathbf{h} . Suppose in addition that, for every $(i, j) \in I_G \times I_R$, there exists at most one path $\rho \in \mathcal{P}_{ij}$ such that $h_\rho > 0$ (if it is the case for one decomposition of \mathbf{T} it has to be the case for all of them). Then we define a *matching* μ of \mathbf{T} as any collection of pairs $(i_1, j_1), \dots, (i_k, j_k)$ such that: $k \geq 2$; $i_\ell \neq i_{\ell'}$ and $j_\ell \neq j_{\ell'}$, for every $\ell \neq \ell'$; $(i_\ell, j_\ell) \in I_G \times I_R$ for every ℓ ; and, for every ℓ , there exists a (necessarily unique) path $\rho^{i_\ell j_\ell} \in \mathcal{P}_{i_\ell j_\ell}$ such that $h_{\rho^{i_\ell j_\ell}} > 0$. In this case, we say that the *support* of μ , denoted by $\text{supp } \mu$, is the list of pairs involved in the paths $\rho^{i_\ell j_\ell}$, for $\ell = 1, \dots, k$.

Then let $S^{ex}(\mathbf{y})$ be the set of transfer plans in $S(\mathbf{y})$ such that: (i) \mathbf{T} has no cycles; (ii) for any flow decomposition \mathbf{h} of \mathbf{T} , and every $(i, j) \in I_G \times I_R$, there exists at most

one path $\rho \in \mathcal{P}_{ij}$ such that $h_\rho > 0$; (iii) for every matching $\mu = (i_1, j_1), \dots, (i_k, j_k)$ of \mathbf{T} , either $\mu' = (i_1, j_2), \dots, (i_{k-1}, j_k)(i_k, j_1)$ is not a matching of \mathbf{T} , or μ and μ' have the same support.

Then we have the following result.

Proposition A.2 *The set $\{\mathbf{T}^\gamma\}_{\gamma \in \mathcal{C}}$ is the set of directions of $S(\mathbf{y})$, and $S^{ex}(\mathbf{y})$ is its set of extreme points. In particular, $S^{ex}(\mathbf{y})$ is a finite set $\{\mathbf{T}^1, \dots, \mathbf{T}^k\}$, and for every matrix $\mathbf{T} \in S(\mathbf{y})$, there exists nonnegative scalars $\lambda_1, \dots, \lambda_k$ such that $\lambda_1 + \dots + \lambda_k = 1$, and nonnegative scalars λ_γ for each $\gamma \in \mathcal{C}$ such that*

$$\mathbf{T} = \sum_{\ell=1}^k \lambda_\ell \mathbf{T}^\ell + \sum_{\gamma \in \mathcal{C}} \lambda_\gamma \mathbf{T}^\gamma.$$

Proof. To see that $\{\mathbf{T}^\gamma\}_{\gamma \in \mathcal{C}}$ is the set of directions of $S(\mathbf{y})$, just note that if $\mathbf{T} \in S(\mathbf{y})$, then, for any $\gamma \in \mathcal{C}$ and any $\lambda > 0$, the transfer plan $\mathbf{T} + \lambda \mathbf{T}^\gamma$ is also in $S(\mathbf{y})$. Furthermore, any flow that is not a cycle, or a combination of cycles, cannot be added to \mathbf{T} without modifying the achieved distribution.

For extreme points, we start by showing that any $\mathbf{T} \in S(\mathbf{y}) \setminus S^{ex}(\mathbf{y})$ can be written as a convex combination of two transfer plans in $S(\mathbf{y})$, and therefore cannot be an extreme point.

First, suppose that \mathbf{T} has a cycle γ , and let $\tau = \min_{(i,j) \in \gamma} t_{ij}$, $\mathbf{T}' = \mathbf{T} - \tau \mathbf{T}^\gamma$ and $\mathbf{T}'' = \mathbf{T} + \tau \mathbf{T}^\gamma$. It is easy to see that \mathbf{T}' and \mathbf{T}'' are both in $S(\mathbf{y})$, and that $\mathbf{T} = \frac{1}{2} \mathbf{T}' + \frac{1}{2} \mathbf{T}''$. Hence we can assume that \mathbf{T} is acyclic. Suppose now that there exists a pair $(i, j) \in I_G \times I_R$ with at least two paths ρ and ρ' in \mathcal{P}_{ij} such that $t_{\ell k} > 0$ for every $(\ell, k) \in \rho$ and every $(\ell, k) \in \rho'$. Then let \mathbf{T}^ρ and $\mathbf{T}^{\rho'}$ be the flows respectively defined by $t_{\ell k}^\rho = \mathbf{1}_{(\ell, k) \in \rho}$ and $t_{\ell k}^{\rho'} = \mathbf{1}_{(\ell, k) \in \rho'}$, and let $\tau = \min_{(\ell, k) \in \rho} t_{\ell k} > 0$ and $\tau' = \min_{(\ell, k) \in \rho'} t_{\ell k} > 0$. We define the new transfer plans

$$\mathbf{T}_1 = \mathbf{T} - \tau \mathbf{T}^\rho + \tau \mathbf{T}^{\rho'},$$

and

$$\mathbf{T}_2 = \mathbf{T} - \tau' \mathbf{T}^{\rho'} + \tau' \mathbf{T}^\rho.$$

It is easy to see that they both achieve \mathbf{y} since they are obtained by reassigning to ρ' some of the money that flows from i to j through ρ , or reciprocally. Furthermore, we have

$$\mathbf{T} = \frac{1/\tau}{1/\tau + 1/\tau'} \mathbf{T}_1 + \frac{1/\tau'}{1/\tau + 1/\tau'} \mathbf{T}_2.$$

Now suppose that \mathbf{T} satisfies properties (i) and (ii) but not (iii) in the definition of $S^{ex}(\mathbf{y})$. Let $\mu = (i_1, j_1), \dots, (i_k, j_k)$ and $\mu' = (i_1, j_2), \dots, (i_{k-1}, j_k)(i_k, j_1)$ be two matchings of \mathbf{T} with different supports. Then let \mathbf{T}^μ and $\mathbf{T}^{\mu'}$ be the flows defined respectively by $t_{ij}^\mu = \mathbf{1}_{(i,j) \in \text{supp } \mu}$, and $t_{ij}^{\mu'} = \mathbf{1}_{(i,j) \in \text{supp } \mu'}$. Because μ and μ' have different supports, we have $\mathbf{T}^\mu \neq \mathbf{T}^{\mu'}$. Let $\tau = \min_{(i,j) \in \text{supp } \mu} t_{ij} > 0$ and $\tau' = \min_{(i,j) \in \text{supp } \mu'} t_{ij} > 0$. Consider the new transfer plans

$$\mathbf{T}_1 = \mathbf{T} - \tau \mathbf{T}^\mu + \tau \mathbf{T}^{\mu'},$$

and

$$\mathbf{T}_2 = \mathbf{T} - \tau' \mathbf{T}^{\mu'} + \tau' \mathbf{T}^\mu.$$

It is easy to see that they both achieve \mathbf{y} since they are only obtained by reassigning to μ' some of the money that flows from sources i_1, \dots, i_k to the sinks j_1, \dots, j_k through μ , and in this reassignment, each sink gains τ from one source and loses τ from another, while each source gives an additional τ to one sink, and reduces its transfer to another source by τ (or reciprocally for \mathbf{T}_2). Furthermore, we have

$$\mathbf{T} = \frac{1/\tau}{1/\tau + 1/\tau'} \mathbf{T}_1 + \frac{1/\tau'}{1/\tau + 1/\tau'} \mathbf{T}_2.$$

Therefore, all extreme points are in $S^{ex}(\mathbf{y})$. Now, let $\mathbf{T} \in S^{ex}(\mathbf{y})$, and suppose that it is not an extreme point. Because all extreme points are in $S^{ex}(\mathbf{y})$ we can write \mathbf{T} as a convex combination of extreme points $\mathbf{T}^1, \dots, \mathbf{T}^k$, all in $S^{ex}(\mathbf{y})$ (we do not need cycles because \mathbf{T} is acyclic). Let $\lambda_\ell > 0$ be the weight of each \mathbf{T}^ℓ in this decomposition. For each $\ell = 1, \dots, k$, and each pair $(i, j) \in I_G \times I_R$, let $\rho_\ell^{ij} \in \mathcal{P}_{ij}$ be the unique path between i and j with positive flow in \mathbf{T}^ℓ . There may be no such path for some ℓ , but if ρ_ℓ^{ij} and $\rho_{\ell'}^{ij}$ exist both, then we must have $\rho_\ell^{ij} = \rho_{\ell'}^{ij}$, for otherwise \mathbf{T} would put a positive flow on both

paths which is impossible.

We pick two of these transfer plans \mathbf{T}^1 and \mathbf{T}^2 , and corresponding flow decompositions \mathbf{h}^1 and \mathbf{h}^2 . For every pair $(i, j) \in I_G \times I_R$, let ρ^{ij} be the unique path between i and j with a positive flow from at least one of the two transfer plans. Let $\tau_{ij}^1 = h_{\rho^{ij}}^1$ and $\tau_{ij}^2 = h_{\rho^{ij}}^2$ be the flows of \mathbf{T}^1 and \mathbf{T}^2 over this path. If a transfer plan has no flow between i and j , we set the corresponding τ_{ij} to 0. We will say that (i, j) is a blue pair if $\tau_{ij}^1 > \tau_{ij}^2$, and a green pair if $\tau_{ij}^1 < \tau_{ij}^2$. Next, consider the following procedure.

First, pick a blue pair (i_1, j_1) . There must exist such a pair for otherwise, $\mathbf{T}^1 = \mathbf{T}^2$. Because i_1 is sending more money over the corresponding path in \mathbf{T}_1 than in \mathbf{T}_2 , there must exist an agent $j_2 \in I_R$ to whom i_1 is sending more money through the path $\rho^{i_1 j_2}$ in \mathbf{T}_2 than in \mathbf{T}_1 . That is (i_1, j_2) is a green pair. For obvious reasons $j_1 \neq j_2$. But then, j_2 is receiving more money from i_1 in \mathbf{T}_2 than in \mathbf{T}_1 . Hence, there must exist an agent $i_2 \in I_G$ such that j_2 receives more money from i_2 in \mathbf{T}_1 than in \mathbf{T}_2 . That is (i_2, j_2) is a blue pair. At this point we can build a new green pair (i_2, j_3) , but it could be the case that $j_3 = j_1$. If this is the case, we stop the construction, and otherwise we continue in this way. Because there is a finite number of agents, we must end up creating a new pair such that one of the agents involved was already part of a previous pair. The construction stops the first time this happens.

This procedure creates an undirected cycle of alternate blue and green pairs. It may be the case that some pairs in the construction are not part of the cycle, in this case, we keep only the cycle. We relabel the blue pairs in the cycle $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$. Then the green pairs are $(i_1, j_2), \dots, (i_{k-1}, j_k), (i_k, j_1)$. The blue pairs form a matching μ for \mathbf{T} . Indeed, since \mathbf{T} puts a positive weight λ_1 on \mathbf{T}_1 , and \mathbf{T}_1 has a positive flow over each path corresponding to a blue pair, \mathbf{T} must also have a positive flow over these pairs. But the green pairs form a matching μ' for \mathbf{T} as well since \mathbf{T}_2 has a positive flow over each path corresponding to a green pair, and \mathbf{T} puts a positive weight on \mathbf{T}_2 .

If μ and μ' have the same support, then we can drop the pairs involved in the cycle and do the construction above again with the remaining pairs. At some point, we must end up with two matchings μ and μ' with different support, for otherwise \mathbf{T}_1 and \mathbf{T}_2 would be

equal. But then, \mathbf{T} violates condition (iii) in the definition of $S^{ex}(\mathbf{y})$.

The remaining of the proposition is just the classical decomposition of elements of a convex polytope. (See for example Rockafellar, 1972, section 19). ■

We now uncover some properties of the set $S^*(\mathbf{y})$ of solutions to (MCF). We start by introducing the notion of cyclical monotonicity. For intuition, suppose that you are currently transferring one dollar from agent i_1 to agent j_1 at a cost $\hat{c}_{i_1 j_1}$ (hence you are using the shortest path between these two agents), and another dollar from i_2 to j_2 at a cost $\hat{c}_{i_2 j_2}$. The total cost of this redistribution is therefore $\hat{c}_{i_1 j_1} + \hat{c}_{i_2 j_2}$. Another way to achieve the same redistribution, however, would be to transfer one dollar from i_1 to j_2 , and one dollar from i_2 to j_1 . If it is the case that

$$\hat{c}_{i_1 j_2} + \hat{c}_{i_2 j_1} < \hat{c}_{i_1 j_1} + \hat{c}_{i_2 j_2},$$

then, clearly, the first plan is not optimal.

Definition 1 (Cyclical Monotonicity) *We say that a subset $\Gamma \subseteq I_G \times I_R$ is $\hat{\mathbf{c}}$ -cyclically monotone if for every sequence $(i_1, j_1), \dots, (i_k, j_k)$ of points in Γ such that all sources and all sinks are distinct, we have*

$$\sum_{\ell=1}^k \hat{c}_{i_\ell j_\ell} \leq \sum_{\ell=1}^k \hat{c}_{i_\ell j_{\ell+1}}$$

with the convention $j_{k+1} = j_1$.

Now, let Γ be the subset of $I_G \times I_R$ such that a pair (i, j) belongs to Γ if there exists an optimal transfer plan $T \in S^*(\mathbf{y})$, a flow decomposition \mathbf{h} of \mathbf{T} (we know that \mathbf{T} is acyclic), and a shortest path ρ from i to j such that $h_\rho > 0$ (we know that all positive path flows of \mathbf{T} are on shortest paths).

Proposition A.3 Γ is $\hat{\mathbf{c}}$ -cyclically monotone.

Proof. Suppose otherwise, and let $(i_1, j_1), \dots, (i_k, j_k)$ be a collection of points over which the monotonicity condition fails. For each of the pairs $\ell = 1, \dots, k$, let \mathbf{T}^ℓ be an optimal transfer plan with flow decomposition \mathbf{h}^ℓ that is positive on a shortest path from i_ℓ to j_ℓ .

Then $\mathbf{T} = \frac{1}{k}\mathbf{T}^1 + \dots + \frac{1}{k}\mathbf{T}^k$ is also an optimal transfer plan whose flow decomposition $\mathbf{h} = \frac{1}{k}\sum_{\ell=1}^k \mathbf{h}^\ell$ has a positive flow between all these pairs. Let \underline{h} be the minimum flow intensity across all pairs. Now consider reassigning \underline{h} from each pair (i_ℓ, j_ℓ) to the pair $(i_\ell, j_{\ell+1})$. This reassignment does not change the final distribution and it leads to a cost reduction of

$$\underline{h} \times \left(\sum_{\ell=1}^k \hat{c}_{i_\ell j_{\ell+1}} - \sum_{\ell=1}^k \hat{c}_{i_\ell j_\ell} \right) > 0,$$

a contradiction to the optimality of \mathbf{T} . ■

A natural corollary of this result is the following.

Corollary A.1 *Let $(i_1, j_1), \dots, (i_k, j_k)$ be a sequence of points in Γ , and suppose that, for every $\ell = 1, \dots, k$, $(i_\ell, j_{\ell+1})$ is also in Γ (with the convention $j_{k+1} = j_1$). Then*

$$\sum_{\ell=1}^k \hat{c}_{i_\ell j_\ell} = \sum_{\ell=1}^k \hat{c}_{i_\ell j_{\ell+1}}$$

Proof. By the cyclical monotonicity inequality, the left-hand side is smaller than the right-hand side. But by rearranging the order of the pairs, we can also obtain the reverse inequality as a cyclical monotonicity inequality. ■

Equipped with this set of results, we can state a sufficient condition for the uniqueness of the cost minimizing transfer plan.

Proposition A.4 (Uniqueness) *Suppose that the cost vector \mathbf{c} and the target distribution \mathbf{y} satisfy the following properties: (a) for every $(i, j) \in I_G \times I_R$, there is a unique shortest path in \mathcal{P}_{ij} ; and (b) for every sequence $(i_1, j_1), \dots, (i_k, j_k)$ of points in Γ such that all sources and all sinks are distinct, we have either $\sum_{\ell=1}^k \hat{c}_{i_\ell j_\ell} < \sum_{\ell=1}^k \hat{c}_{i_\ell j_{\ell+1}}$, or the list of arcs in the shortest paths $\rho^{i_1, j_1}, \dots, \rho^{i_k, j_k}$ and the list of arcs in the shortest paths $\rho^{i_1, j_2}, \dots, \rho^{i_k, j_1}$ are the same. Then $S^*(\mathbf{y})$ is a singleton.*

Proof. Suppose that $S^*(\mathbf{y})$ is not a singleton. Then there exists a transfer plan \mathbf{T} in $S^*(\mathbf{y})$ that is not in $S^{ex}(\mathbf{y})$. Since \mathbf{T} must be acyclic, it must fail property (ii) or (iii) of

the definition of $S^{ex}(\mathbf{y})$. Suppose first that it fails property (ii). Then there must exist a pair $(i, j) \in I_G \times I_R$ and a flow decomposition \mathbf{h} of \mathbf{T} with positive flows on two distinct paths $\rho \neq \rho'$ of \mathcal{P}_{ij} . By Theorem A.1, both of these paths must be shortest paths, but that contradicts (a). Suppose now that it satisfies property (ii) but fails (iii). Then let $\mu = (i_1, j_1), \dots, (i_k, j_k)$ and $\mu' = (i_1, j_2), \dots, (i_k, j_1)$ be two matchings of \mathbf{T} with different support such that a flow decomposition \mathbf{h} of \mathbf{T} puts positive flows on μ and μ' . Then, by Corollary A.1, we must have $\sum_{\ell=1}^k \hat{c}_{i_\ell j_\ell} = \sum_{\ell=1}^k \hat{c}_{i_\ell j_{\ell+1}}$, which contradicts (b). ■

This allows us to prove the following generic uniqueness result.

Proposition A.5 (Generic Uniqueness) *Generically in \mathbf{c} , the cost minimizing transfer plan is unique for every \mathbf{y}^0 and every feasible \mathbf{y} .*

Proof. Consider the set of cost vectors \tilde{C} such that: (i) for every pair of agents (i, j) , and every pair of distinct paths $\rho \neq \rho'$ in \mathcal{P}_{ij} , $c_\rho \neq c_{\rho'}$; and (ii), for every sequence $(i_1, j_1), \dots, (i_k, j_k)$ of arcs in A , such that $(i_\ell, j_{\ell+1}) \in A$ for each ℓ , and every choice of paths $\rho_\ell \in \mathcal{P}_{i_\ell j_\ell}$, and $\rho'_\ell \in \mathcal{P}_{i_\ell j_{\ell+1}}$, such that the list of arcs in the paths $(\rho)_{\ell=1, \dots, k}$ and $(\rho')_{\ell=1, \dots, k}$ are distinct, then $\sum_{\ell=1}^k c_{\rho_\ell} \neq \sum_{\ell=1}^k c_{\rho'_\ell}$.

Note that we need to assume that the list of arcs in ρ and ρ' are distinct, for otherwise the two sums are necessarily equal. It is easy that any cost vector in \tilde{C} satisfies properties (a) and (b) of proposition A.4 for any $\mathbf{y} \in Y$. But the set of cost vectors \mathbf{c} that do not belong to \tilde{C} is a finite reunion of hyperplanes defined by a linear inequality, therefore \tilde{C} is generic in the set of possible cost vectors. ■

Before going back to the original problem, we provide some results on the value function of the cost minimization problem. As usual in linear programming, the minimization program has a dual maximization program. In this case, it can be written as a program over a vector of “prices” ϕ in \mathbb{R}^N . For any feasible \mathbf{y} ,

$$c(\mathbf{y}, \mathbf{y}^0) = \max_{\phi} \sum_{i=1}^N \phi_i (y_i - y_i^0) \quad \text{s.t.} \quad \phi_j - \phi_i \leq c_{ij} \quad \forall i \neq j.$$

As a consequence, we have the following result.

Proposition A.6 *The cost function $c(\mathbf{y}, \mathbf{y}^0)$ is convex in \mathbf{y} and \mathbf{y}^0 , supermodular in $\mathbf{y} - \mathbf{y}^0$, concave in \mathbf{c} and continuous in all variables. Furthermore, it depends on \mathbf{c} only through $\hat{\mathbf{c}}$. $S^*(\mathbf{y})$ is upper hemicontinuous in \mathbf{y} , \mathbf{y}^0 and \mathbf{c} .*

Proof. In the original problem, we are minimizing an objective function that is linear in $\hat{\mathbf{c}}$ over a convex set. In the dual problem, we are maximizing an objective function that is linear in \mathbf{y} and \mathbf{y}^0 over a convex set. The dual formulation of the problem maximizes a supermodular function in $(\mathbf{y} - \mathbf{y}^0, \phi)$ over a lattice, therefore its value function is supermodular. The continuity properties can be derived by applying the maximum theorem (see, for example, Aliprantis and Border, 2006) to the minimization problem after having reduced the space over which the function is minimized to the compact set of acyclic transfers that achieve \mathbf{y} . The fact that the value function depends only on $\hat{\mathbf{c}}$ is a direct consequence of Theorem A.1. ■

Going back to the original problem, we can now rewrite the problem of maximizing the potential, as $\max_{\mathbf{y} \in Y} \sum_{i=1}^n U_i(y_i) - c(\mathbf{y}, \mathbf{y}^0)$, and summarize our results in the following theorem.

Theorem A.2 *There is a unique equilibrium distribution \mathbf{y}^* . It is continuous as a function of \mathbf{y}^0 and \mathbf{c} , and depends on \mathbf{c} only through $\hat{\mathbf{c}}$. The set of Nash equilibria of the transfer game is a nonempty, compact and convex polytope given by*

$$S^* = \arg \min_{\mathbf{T}} \sum_{1 \leq i, j \leq N} c_{ij} t_{ij} \quad \text{s.t.} \quad \sum_{i \neq j} (t_{ji} - t_{ij}) = y_i^* - y_i^0 \quad \forall i.$$

It is generically a singleton. Furthermore, every transfer network in S^ is acyclic, and all its positive flows are on shortest paths. As a correspondence, S^* is upper hemicontinuous in $(\mathbf{y}^0, \mathbf{c})$, and depends on \mathbf{c} only through $\hat{\mathbf{c}}$.*

Proof. Most points are direct consequences of our results on the cost minimization problem. The only point that needs proof is the existence, uniqueness and continuity of the solution to the first program. Note first that Y is closed, and bounded since no agent can get more than $\sum_i y_i^0$. The objective function is continuous, by assumption for the first

term, and as a consequence of convexity for the cost term. That gives us existence. The program is strictly concave in \mathbf{y} , by strict concavity of the $U_i(\cdot)$ functions, and convexity of $c(\cdot, \mathbf{y}^0)$, therefore the solution is unique. Furthermore, the maximum theorem implies that the solution to the problem is continuous in \mathbf{y}^0 and \mathbf{c} . ■

APPENDIX B : Best response dynamics

In this section, we prove Proposition 2. As a preliminary, we show a few lemmas.

Lemma B.2 *For every scalar λ , the set $\Phi_\lambda = \{\mathbf{T} : \varphi(\mathbf{T}) \geq \lambda\}$ is compact.*

Proof. We know that $\varphi(\cdot)$ attains its maximum over the set of transfers. Let $\bar{\varphi}$ denote the value of this maximum. The set Φ_λ is the reciprocal image of the interval $[\lambda, \bar{\varphi}]$ by the continuous function $\varphi(\cdot)$, hence it is closed (if $\lambda > \bar{\varphi}$, then Φ_λ is empty, and the lemma holds vacuously). Suppose, by contradiction, that it is unbounded, and let $\{\mathbf{T}^n\}$ be an unbounded sequence of transfers in Φ_λ , so that $\|\mathbf{T}^n\| \rightarrow \infty$. Fix a scalar $K \geq 0$. We can assume that, for every n , $\|\mathbf{T}^* - \mathbf{T}^n\| \geq K$. Let \mathbf{T}^* be a maximizer of $\varphi(\cdot)$. Clearly, $\mathbf{T}^* \in \Phi_\lambda$. Consider the sequence $\tilde{\mathbf{T}}^n$ defined by

$$\tilde{\mathbf{T}}^n = \frac{K}{\|\mathbf{T}^* - \mathbf{T}^n\|} \mathbf{T}^n + \frac{\|\mathbf{T}^* - \mathbf{T}^n\| - K}{\|\mathbf{T}^* - \mathbf{T}^n\|} \mathbf{T}^*$$

Note that, for every n , $\|\tilde{\mathbf{T}}^n - \mathbf{T}^*\| = K$. By concavity of φ , we have

$$\begin{aligned} \bar{\varphi} \geq \varphi(\tilde{\mathbf{T}}^n) &\geq \frac{K}{\|\mathbf{T}^* - \mathbf{T}^n\|} \varphi(\mathbf{T}^n) + \frac{\|\mathbf{T}^* - \mathbf{T}^n\| - K}{\|\mathbf{T}^* - \mathbf{T}^n\|} \varphi(\mathbf{T}^*) \\ &\geq \frac{K}{\|\mathbf{T}^* - \mathbf{T}^n\|} \lambda + \frac{\|\mathbf{T}^* - \mathbf{T}^n\| - K}{\|\mathbf{T}^* - \mathbf{T}^n\|} \bar{\varphi} \xrightarrow{n \rightarrow \infty} \bar{\varphi} \end{aligned}$$

Hence the sequence $\varphi(\tilde{\mathbf{T}}^n)$ converges to $\bar{\varphi}$. Since the sequence $\tilde{\mathbf{T}}^n$ lies in the compact set of points at distance K of \mathbf{T}^* , it has a converging subsequence. Let $\tilde{\mathbf{T}}^\infty$ denote the limit of this subsequence. It is at distance K of \mathbf{T}^* . By continuity of φ , we must have $\varphi(\tilde{\mathbf{T}}^\infty) = \bar{\varphi}$. Therefore, we have found a maximizer of φ at distance K of \mathbf{T}^* . Since we can do so for

every K , this implies that the set S^* is unbounded, a contradiction since it is compact by theorem A.2. ■

For a player i , and a transfer profile \mathbf{T} , let $BR_i(\mathbf{T})$ be the set of pairs $(\mathbf{T}'_i, \mathbf{T}_{-i})$ such that \mathbf{T}'_i is a best response to \mathbf{T}_{-i} . For any ordering (a permutation) σ of players let

$$BR^\sigma(\mathbf{T}) = BR_{\sigma(n)} \circ BR_{\sigma(n-1)} \circ \cdots \circ BR_{\sigma(1)}(\mathbf{T}).$$

Lemma B.3 *For any ordering σ , the best response correspondence BR^σ is nonempty, compact-valued and upper hemicontinuous. Furthermore, the set of fixed points of BR^σ is exactly the set S^* of Nash equilibria of the transfer game.*

Proof. First consider $BR_i(\cdot)$. For every $\mathbf{T}' \in BR_i(\mathbf{T})$, we have $\mathbf{T}'_{-i} = \mathbf{T}_{-i}$, hence the correspondence is continuous in all dimensions $j \neq i$. On dimension i , we have, by the best response potential property,

$$\mathbf{T}'_i \in \arg \max_{\hat{\mathbf{T}}_i \in \mathbb{R}_+^{n-1}} \varphi(\hat{\mathbf{T}}_i, \mathbf{T}_{-i})$$

When solving this program, we can fix a transfer plan \mathbf{T}_i^0 for player i , and restrict the program above to transfers in the set

$$\mathcal{S}(\mathbf{T}_{-i}) = \{\hat{\mathbf{T}}_i : (\hat{\mathbf{T}}_i, \mathbf{T}_{-i}) \in \Phi_{\varphi(\mathbf{T}_i^0, \mathbf{T}_{-i})}\},$$

which is compact by lemma B.2. It is also easy to see that the correspondence $\mathcal{S}(\cdot)$ is continuous. By continuity of $\varphi(\cdot)$, we can apply the maximum theorem to conclude that the maximizer correspondence of this program is nonempty, compact-valued and upper hemicontinuous. This implies that $BR_i(\cdot)$ is nonempty, compact-valued and upper hemicontinuous, for every i , and therefore that $BR^\sigma(\cdot)$ satisfies these properties as well.

If \mathbf{T} is a Nash equilibrium of the transfer game, it is clearly a fixed point of BR^σ . Suppose that $\mathbf{T} \in BR^\sigma(\mathbf{T})$, then φ remains constant along the sequence of best replies that lead to $BR^\sigma(\mathbf{T})$. This implies that, at each step i of this sequence, \mathbf{T} is among the best replies of player i . Therefore \mathbf{T} is a Nash equilibrium of the transfer game. ■

We say that $\{\mathbf{T}^k\}$ is a best-response dynamics sequence if, for every k ,

$$\mathbf{T}^k \in \underbrace{BR^\sigma \circ \dots \circ BR^\sigma}_{k \text{ times}}(\mathbf{T}^0).$$

Proposition B.7 *The limit set of any best-response dynamics sequence is a subset of the set of Nash equilibria S^* . For any best-response dynamics sequence $\{\mathbf{T}^k\}$, the sequence of corresponding consumptions $\{\mathbf{y}^k\}$ converges to the unique equilibrium distribution \mathbf{y} .*

Proof. Pick any best-response dynamics sequence $\{\mathbf{T}^k\}$. First note that the sequence $\varphi(\mathbf{T}^k)$ is increasing and bounded above by $\bar{\varphi}$, and therefore converges. We denote its limit by φ_∞ . Since, for every k , $\varphi(\mathbf{T}^k) \geq \varphi(\mathbf{T}^0)$, the sequence \mathbf{T}^k lies in the compact set $\Phi_{\varphi(\mathbf{T}^0)}$. Hence $\{\mathbf{T}^k\}$ admits a converging subsequence. Let $\{\mathbf{T}^{g(k)}\}$ be such a subsequence, and $\mathbf{T}^{g(\infty)}$ its limit. By continuity of φ , we have $\varphi(\mathbf{T}^{g(\infty)}) = \varphi_\infty$.

Consider the subsequence $\mathbf{T}^{g(k)+1}$. Since it lies in a compact set, we can extract a converging subsequence from this new sequence. Assume, without loss of generality for the argument to follow, that $\mathbf{T}^{g(k)+1}$ is itself convergent, and denote its limit by $\tilde{\mathbf{T}}$. Since $\mathbf{T}^{g(k)+1} \in BR^\sigma(\mathbf{T}^{g(k)})$, we have by upper hemicontinuity of BR^σ , that $\tilde{\mathbf{T}} \in BR^\sigma(\mathbf{T}^{g(\infty)})$. Since $\mathbf{T}^{g(k)+1}$ is also a converging subsequence of \mathbf{T}^k , we have $\varphi(\tilde{\mathbf{T}}) = \varphi_\infty = \varphi(\mathbf{T}^{g(\infty)})$. This implies that $\mathbf{T}^{g(\infty)}$ is also in $BR^\sigma(\mathbf{T}^{g(\infty)})$, and is therefore in S^* . This must hold for any limit of a converging subsequence of \mathbf{T}^k , so the limit set of \mathbf{T}^k is a subset of S^* .

Next consider the sequence of consumptions \mathbf{y}^k . Since $\{\mathbf{T}^k\}$ lies in the compact set $\Phi_{\varphi(\mathbf{T}^0)}$, and since the function that maps a transfer profile to the corresponding distribution is continuous, it is uniformly continuous on $\Phi_{\varphi(\mathbf{T}^0)}$. Pick $\varepsilon > 0$, and $\delta > 0$, such that for every \mathbf{T} and \mathbf{T}' in $\Phi_{\varphi(\mathbf{T}^0)}$, $\|\mathbf{y} - \mathbf{y}'\| < \delta$. By definition of the limit set, we can pick K , such that for every $k > K$, \mathbf{T}^k is within distance δ of the limit set S^* of the sequence. Then, for every $k > K$, \mathbf{y}^k is within distance ε of the distribution \mathbf{y} associated with some $\mathbf{T} \in S^*$, that is, $\|\mathbf{y}^k - \mathbf{y}^*\| < \varepsilon$. Hence \mathbf{y}^k converges to the unique equilibrium distribution \mathbf{y} .

■

REFERENCES

Aliprantis, Charalambos D. and Border, Kim C. 2006. “Infinite Dimensional Analysis. A Hitchhiker’s Guide.” 3rd Ed, *Springer*, Berlin Heidelberg New York.

Galichon, Alfred. 2011. “Theoretical and Empirical Aspects of Matching Markets.” *Teaching Notes*, Columbia University.

Rockafellar, R. Tyler. 1972. “Convex Analysis.” Princeton University Press, Princeton, NY.